

OCTAHEDRALITY IN LIPSCHITZ FREE BANACH SPACES

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ABSTRACT. The aim of this note is to study octahedrality in vector valued Lipschitz-free Banach spaces on a metric space under topological hypotheses on it by analysing the weak-star strong diameter two property in Lipschitz functions spaces. Also, we show an example which proves that our results are optimal and that octahedrality in vector-valued Lipschitz-free Banach spaces actually relies on the underlying metric space as well as on the Banach one.

1. INTRODUCTION

Lipschitz functions spaces (denoted by $Lip(M)$) and its preduals [25], Lipschitz-free Banach spaces (denoted by $\mathcal{F}(M)$), have been recently studied under a topological point of view (e.g. [10], [15], [19]). Geometrical properties in such spaces have been also considered, as Daugavet property. Indeed, Daugavet property in Lipschitz functions spaces has been characterized in [17] in terms of “locality” in the compact case and provides examples of metric spaces whose free-Lipschitz Banach space has an octahedral norm. On the other hand, in [9] it has been recently proved that given M an infinite metric space then the free space $\mathcal{F}(M)$ contains a complemented copy of ℓ_1 and, consequently, $\mathcal{F}(M)$ has an equivalent norm which is an octahedral norm [14].

Motivated by this kind of results, the aim of this note is to go further and analyse octahedrality in vector-valued free-Lipschitz Banach spaces. Indeed, we introduce the Banach space of vector-valued Lipschitz-free Banach space which, up the best of our knowledge, has not been previously considered and, in Section 2, we prove in Theorem 2.4 that such spaces have an octahedral norm whenever their underlying metric space satisfies some topological assumptions such as being unbounded or not being uniformly discrete and a condition of existence of extension of vector-valued Lipschitz functions (see Definition 2.3). Consequently, such Banach spaces can not have any point of Fréchet differentiability. Moreover, we will exhibit an example of a metric space such that, depending on the underlying Banach space, the geometry of vector valued Lipschitz-free Banach space changes its behaviour from

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having a point of Fréchet differentiability to having an octahedral norm. This will have two important consequences: on the one hand, as there are vector-valued Lipschitz-free Banach spaces which contain points of Fréchet differentiability, we prove that our results on octahedrality are optimal; on the other hand, this proves that the geometry of vector-valued Lipschitz-free Banach spaces is determined by the underlying metric space as well as by the target Banach space. We will end by exhibiting some consequences of Theorem 2.4 and open problems in Section 3

We shall now introduce some notation. We will consider only real Banach spaces. Given X a Banach space, B_X (respectively S_X) stands for the closed unit ball (respectively the unit sphere) of X . Given a Banach space X , we will mean by a slice of B_X a subset of the following form

$$S(B_X, f, \alpha) := \{x \in B_X : f(x) > 1 - \alpha\},$$

where $f \in S_{X^*}$ and $\alpha > 0$. If X is a dual space, say $X = Y^*$, by a weak-star slice of B_{X^*} we will mean a slice $S(B_X, y, \alpha)$ where $y \in Y$.

Given M a metric space with a designated origin 0 and a Banach space X , we will denote by $Lip(M, X)$ the Banach space of all X -valued Lipschitz function on M which vanish at 0 under the standard Lipschitz norm

$$\|f\| := \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} \mid x, y \in M, x \neq y \right\}.$$

First of all, notice that we can consider every point of M as an origin with no loss of generality. Indeed, given $x, y \in M$, let $Lip_x(M, X)$ ($Lip_y(M, X)$) be the space of X -valued Lipschitz functions which vanish at x (respectively at y). Then the map

$$\begin{array}{ccc} Lip_x(M, X) & \longrightarrow & Lip_y(M, X) \\ f & \longmapsto & f - f(y), \end{array}$$

defines an onto linear isometry. So the designated origin will be freely chosen.

From a straightforward application of Ascoli-Arzelà theorem it can be checked that $B_{Lip(M, X^*)}$ is a compact set for the pointwise topology. Hence $Lip(M, X^*)$ is itself a dual Banach space. In fact, the map

$$\begin{array}{ccc} \delta_{m,x} : Lip(M, X^*) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & f(m)(x) \end{array}$$

defines a linear and bounded map for each $m \in M$ and $x \in X$. In other words, $\delta_{m,x} \in Lip(M, X^*)^*$. Then if we define

$$\mathcal{F}(M, X) := \overline{\text{span}}(\{\delta_{m,x} \mid m \in M, x \in X\})$$

then we have that $\mathcal{F}(M, X)^* = Lip(M, X^*)$. Furthermore, a bounded net $\{f_s\}$ in $Lip(M, X^*)$ converges in the weak-star topology to a function $f \in Lip(M, X^*)$ if, and only if, $\{f_s(m)\} \rightarrow f(m)$ for each $m \in M$, where last convergence is in the weak-star topology of X^* .

Note that given $f : M \longrightarrow X^*$ a Lipschitz map then there exists a linear operator $T_f : \mathcal{F}(M) \longrightarrow X^*$ such that $T_f \circ \delta_m = f(m)$ for each $m \in M$. This map

$$\begin{array}{ccc} \Phi : Lip(M, X^*) & \longrightarrow & L(\mathcal{F}(M), X^*) \\ f & \longmapsto & T_f \end{array}$$

is an isometric isomorphism (see e.g. [18]). Now we have the following

Proposition 1.1. Φ is $w^* - w^*$ continuous.

Proof. Note that Φ is $w^* - w^*$ continuous if, and only if, for every $z \in \mathcal{F}(M) \widehat{\otimes}_\pi X$ one has that $z \circ \Phi$ is a weak-star continuous functional. By [12, Corollary 3.94] it is enough to prove that, given $z \in \mathcal{F}(M) \widehat{\otimes}_\pi X$, we have that $Ker(z \circ \Phi) \cap B_{Lip(M, X^*)}$ is weak-star closed. So, pick $z \in \mathcal{F}(M) \widehat{\otimes}_\pi X$ and consider $\{f_s\}$ a net in $Ker(z \circ \Phi) \cap B_{Lip(M, X^*)}$ which is weak-star convergent to f , and let us prove that $(z \circ \Phi)(f) = 0$. To this aim, pick a positive number $\varepsilon > 0$. Note that z can be expressed as

$$z := \sum_{n=1}^{\infty} \gamma_n \otimes x_n$$

where $\gamma_n \in \mathcal{F}(M)$ and $x_n \in X$ verify that $\sum_{n=1}^{\infty} \|\gamma_n\| \|x_n\| < \infty$ [23, Proposition 2.8]. Now, consider $\{\varepsilon_n\}$ a sequence in \mathbb{R}^+ such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{\varepsilon}{3}$ and consider, for each $n \in \mathbb{N}$, an element $\psi_n \in span\{\delta_m : m \in M\}$ verifying $\|\gamma_n - \psi_n\| \|x_n\| < \frac{\varepsilon_n}{2}$ for each $n \in \mathbb{N}$. As it is clear that $\sum_{n=1}^{\infty} \|\psi_n\| \|x_n\| < \infty$, consider $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} \|\psi_n\| \|x_n\| < \frac{\varepsilon}{6}$. Finally, in view of weak-star topology of $Lip(M, X^*)$, it is obvious that $\{f_s(\psi_n)(x_n)\} \rightarrow f(\psi_n)(x_n)$ for each $n \in \mathbb{N}$, hence we can find s such that $|(f - f_s)(\psi_n)(x_n)| < \frac{\varepsilon}{3k}$ for each $n \in \{1, \dots, k\}$. Now, keeping in mind that $\|f - f_s\| \leq 2$, one has

$$\begin{aligned} |(z \circ \Phi)(f)| &= |(z \circ \Phi)(f - f_s)| = \left| \sum_{n=1}^{\infty} T_{f-f_s}(\gamma_n)(x_n) \right| \leq \\ &\left| \sum_{n=1}^{\infty} T_{f-f_s}(\psi_n)(x_n) \right| + \|f - f_s\| \sum_{n=1}^{\infty} \|\gamma_n - \psi_n\| \|x_n\| \leq \\ &\sum_{n=1}^k |(f - f_s)(\psi_n)(x_n)| + \|f - f_s\| \sum_{n=k+1}^{\infty} \|\psi_n\| \|x_n\| + \frac{\varepsilon}{3} \\ &< \sum_{n=1}^k \frac{\varepsilon}{3k} + \frac{2\varepsilon}{3} = \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary we conclude that $(z \circ \Phi)(f) = 0$, so we are done. ■

The norm on a Banach space X is said to be octahedral if for every $\varepsilon > 0$ and for every finite-dimensional subspace M of X there is some y in the unit sphere of X such that

$$\|x + \lambda y\| \geq (1 - \varepsilon)(\|x\| + |\lambda|)$$

holds for every $x \in M$ and for every scalar λ (see [11]).

We recall that a Banach space X satisfies the slice diameter two property (respectively diameter two property, strong diameter two property) if every slice (respectively nonempty weakly open subset, convex combination of slices) of the closed unit ball has diameter two. If X is itself a dual Banach space then weak-star slice diameter two property (respectively weak-star diameter two property and weak-star strong diameter two property) can be defined as usual, invoking weak-star slices (respectively nonempty weakly-open subset, convex combination of weak-star slices) of the unit ball of X . These property, which are extremely opposite to Radon-Nikodým property, have been deeply studied since last few years. For instance, it has been recently proved [2, 3] that each one of the above properties is different from the rest in an extreme way.

Banach spaces which satisfy some of the diameter two properties are infinite-dimensional uniform algebras [22], Banach spaces satisfying Daugavet property [24], non-reflexive M-embedded spaces [21], etc.

It is known that the norm on a Banach space X is octahedral if, and only if, X^* satisfies the weak-star strong diameter two property [4, Theorem 2.1]. It is also known that if a Banach space X has an octahedral norm, then X contains an isomorphic copy of ℓ_1 [14].

Finally, given a Banach space X and a point $x \in X$, we say that x is a point of Fréchet differentiability if, for each $h \in X$, we have that

$$f'(x)(h) := \lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t},$$

uniformly for $h \in S_X$ and $f'(x) : X \rightarrow \mathbb{R}$ is a continuous and linear functional (see [14]).

It is clear that a Banach space which has an octahedral norm does not have any point of Fréchet differentiability.

2. MAIN RESULTS.

Let M be a metric space and X a Banach space. Notice that we have a useful description of $\mathcal{F}(M, X)$ because we know a dense subspace of it. This fact will play an important role in the following because diameter two properties actually rely on dense subspaces in the following sense.

Proposition 2.1. *Let X be a Banach space. Let $Y \subseteq X^*$ a norm dense subspace. Then:*

- (1) *If for each $f \in S_Y$ and $\alpha \in \mathbb{R}^+$ the slice $S(B_X, f, \alpha)$ has diameter two, then X has the slice diameter two property.*

- (2) If for each $f_1, \dots, f_n \in S_Y$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ such that $W := \bigcap_{i=1}^n S(B_X, f_i, \alpha_i) \neq \emptyset$ it follows that W has diameter two, then X has the diameter two property.
- (3) If for each $f_1, \dots, f_n \in S_Y, \alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ and $\lambda_1, \dots, \lambda_n \in]0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$, the convex combination of slices $\sum_{i=1}^n \lambda_i S(B_X, f_i, \alpha_i)$ has diameter two, then X satisfies the strong diameter two property

Proof. We will prove statement (1), being the proof of (2) and (3) completely similar.

Pick $S := S(B_X, f, \alpha)$ a slice of B_X . As Y is norm dense in X^* we can find $\varphi \in S_Y$ such that $\|f - \varphi\| < \frac{\alpha}{2}$.

By hypothesis, given an arbitrary $\delta \in \mathbb{R}^+$ we can find $x, y \in S(B_X, \varphi, \frac{\alpha}{2})$ such that $\|x - y\| > 2 - \delta$. Let us prove that $x \in S$, being the proof of $y \in S$ similar. Bearing in mind that $\varphi(x) > 1 - \frac{\alpha}{2}$ and that $\|f - \varphi\| < \frac{\alpha}{2}$ we deduce

$$f(x) = \varphi(x) + (f - \varphi)(x) \geq \varphi(x) - \|f - \varphi\| > 1 - \alpha.$$

On the other hand, as $x, y \in S$, we conclude

$$2 - \delta < \|x - y\| \leq \text{diam}(S).$$

As $\delta \in \mathbb{R}^+$ was arbitrary we conclude that X has the slice diameter two property, as desired ■

Now we consider the weak-star version of Proposition above.

Proposition 2.2. *Let X be a Banach space and $Y \subseteq X$ be a dense subspace. Then:*

- (1) If for each $y \in S_Y$ and $\alpha \in \mathbb{R}^+$ the slice $S(B_{X^*}, y, \alpha)$ has diameter two, then X has the weak-star slice diameter two property.
- (2) If for each $y_1, \dots, y_n \in S_Y$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ such that $W := \bigcap_{i=1}^n S(B_{X^*}, y_i, \alpha_i) \neq \emptyset$ one has that W has diameter two, then X has the weak-star diameter two property.
- (3) If for $y_1, \dots, y_n \in S_Y, \alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ and $\lambda_1, \dots, \lambda_n \in]0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$ the convex combination of weak-star slices $\sum_{i=1}^n \lambda_i S(B_{X^*}, y_i, \alpha_i)$ has diameter two, then X satisfies the strong diameter two property

Now we need the following

Definition 2.3. Let M be a metric space and let X be a Banach space.

We will say that the pair (M, X) satisfies the contraction-extension property (CPE) if given $N \subseteq M$ and $f : N \rightarrow X$ a Lipschitz function then there exists $F : M \rightarrow X$ a Lipschitz function which extends to f such that

$$\|F\|_{\text{Lip}(M, X)} = \|f\|_{\text{Lip}(N, X)}.$$

On the one hand note that, in the particular case of being M a Banach space, the definition given above agrees with the one given in [6].

On the other hand, let us give some examples of pairs which have the CPE. First of all, given M a metric space, the pair (M, \mathbb{R}) has the (CPE) [25, Theorem 1.5.6]. In addition, in [6, Chapter 2] we can find some examples of Banach spaces X such that the pair (X, X) satisfies the contraction extension property as Hilbert spaces and ℓ_∞^n . Finally, if Y is a strictly convex Banach space such that there exists a Banach space X with $\dim(X) \geq 2$ and verifying that the pair (X, Y) has the CPE, then Y is a Hilbert space [6, Theorem 2.11].

Let us explain roughly the key idea of the main result, which proves, for every unbounded or not uniformly discrete metric space M , that the norm on $\mathcal{F}(M, X)$ is octahedral, whenever the pair (M, X^*) has the CPE, where X is a Banach space. For this, it is enough to show that every convex combination of w^* -slices C in the unit ball of $Lip(M, X^*)$ has diameter exactly 2. What it is done first is to observe that it is enough to consider w^* -slices given by elements in $span\{\delta_{m,x} / m \in M, x \in X\}$, which is based on Proposition 2.2. Now, depending on the kind of considered metric space, we construct a pair of Lipschitz functions for every w^* -slice of C . Different pairs are defined on different finite metric subspaces so that each pair of these functions have norm preserving extensions to Lipschitz functions in a w^* -slice of C from the CPE assumption, and we control the norm of each pair only on a finite metric space, so that the difference between the elements of every pair is also controlled. Now the estimates for the extensions are possible and in this way we get that C has diameter 2. Of course, there are details which depend on the kind of considered metric space, but the existence of the above unified idea motivated to us to show the following result in a joint way.

Theorem 2.4. *Let M be an infinite pointed metric space and let X be a Banach space. Assume that the pair (M, X^*) has the CPE. If M is unbounded or is not uniformly discrete then the norm on $\mathcal{F}(M, X)$ is octahedral. Consequently, the unit ball of $\mathcal{F}(M, X)$ can not have any point of Fréchet differentiability.*

Proof. We will prove that $Lip(M, X^*)$ has the weak-star strong diameter two property, which is equivalent to the thesis of the Theorem. Let $C = \sum_{i=1}^k \lambda_i S(B_{Lip(M, X^*)}, \varphi_i, \alpha)$ be a convex combination of weak-star slices in $Lip(M, X^*)$ and let us prove that C has diameter exactly 2. From Proposition 2.1 we can assume that $\varphi_i \in span\{\delta_{m,x} / m \in M, x \in X\}$ for each $i \in \{1, \dots, k\}$. So assume that

$$\varphi_i = \sum_{j=1}^{n_i} \lambda_j^i \delta_{m_{i,j}, x_{i,j}},$$

for suitable $n_i \in \mathbb{N}$, $m_{i,j} \in M \setminus \{0\}$, $x_{i,j} \in X$, $\lambda_j^i \in \mathbb{R}$ for $i \in \{1, \dots, k\}$, $j \in \{1, \dots, n_i\}$.

Pick $g_i \in S(B_{Lip(M, X^*)}, \varphi_i, \alpha)$ and $\delta_0 \in \mathbb{R}^+$ verifying

$$0 < \delta < \delta_0 \Rightarrow \frac{\varphi_i(g_i)}{1 + \delta} > 1 - \alpha \quad \forall i \in \{1, \dots, k\}.$$

Fix $0 < \delta < \delta_0$.

Now, in a first step we will define for every $i \in \{1, \dots, k\}$ a subspace $M_i \subset M$ and functions F_i and G_i in $Lip(M_i, X^*)$.

We will do this depending on following cases: M is unbounded, M is bounded, discrete but not uniformly discrete or M is bounded and $0 \in M'$. It is clear that each one of these cases holds whenever M is unbounded or not uniformly discrete and that all together cover the assumption of the Theorem.

Assume that M is unbounded. Then there exists $\{m_n\} \subseteq M$ verifying

$$\{d(m_n, 0)\} \rightarrow \infty.$$

Hence

$$\{d(m_n, m)\} \rightarrow \infty$$

for each $m \in M$ in view of triangle inequality. Now pick an positive integer N so that

$$(2.1) \quad \frac{d(m_{i,j}, 0)}{d(m_N, m_{i,j})} + \frac{\|g_i(m_{i,j})\|}{d(m_N, m_{i,j})} < \delta \quad \forall i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}.$$

Choose $x^* \in S_{X^*}$ and define $M_i := F := \{0\} \cup \bigcup_{i=1}^k \bigcup_{j=1}^{n_i} \{m_{i,j}\} \cup \{m_N\}$ for every $1 \leq i \leq k$. (In this case M_i does not depend on i). Also, we define $F_i, G_i : M_i \rightarrow X^*$ given by

$$\begin{aligned} F_i(m_{i,j}) &= G_i(m_{i,j}) = g_i(m_{i,j}) \quad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}, \\ F_i(0) &= G_i(0) = 0, F_i(m_N) = -G_i(m_N) = d(m_N, 0)x^*. \end{aligned}$$

Assume now that M is bounded and discrete, but not uniformly discrete. As M is discrete we can find $r > 0$ such that

$$B(0, r) = \{0\}, B(m_{i,j}, r) = \{m_{i,j}\} \quad \forall i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}.$$

Also, as M is not uniformly discrete we can find $\{x_n\}, \{y_n\}$ a pair of sequences in M such that $0 < d(x_n, y_n) \rightarrow 0$. Pick $n \in \mathbb{N}$ satisfying $d(x_n, y_n) < \delta$ and so that

$$(2.2) \quad \frac{1 + \frac{d(x_n, y_n)}{d(x_n, v)}}{1 - \frac{d(x_n, y_n)}{d(x_n, v)}} < 1 + \delta \quad \forall v \in \{m_{i,j} : 1 \leq i \leq k, 1 \leq j \leq n_i\} \cup \{0\}.$$

Note that such an n exists since $\{d(x_n, v)^{-1}\}$ is a well defined bounded sequence because M is discrete and bounded in this case. Given $i \in \{1, \dots, k\}$ and $x^* \in S_{X^*}$ define $M_i := \{0\} \cup \bigcup_{j=1}^{n_i} \{m_{i,j}\} \cup \{x_n, y_n\}$ and $F_i, G_i : M_i \rightarrow \mathbb{R}$ given by

$$F_i(0) = g_i(0) = 0, F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}) \quad \forall j \in \{1, \dots, n_i\},$$

and

$$\begin{aligned} F_i(x_n) &= G_i(x_n) = g_i(x_n), F_i(y_n) = g_i(x_n) + d(y_n, x_n)x^*, \\ G_i(y_n) &= g_i(x_n) - d(y_n, x_n)x^*. \end{aligned}$$

Finally, we assume that M is bounded and $0 \in M'$. Then we can find $\{m_n\}$ a sequence in $M \setminus \{0\}$ such that $\{m_n\} \rightarrow 0$. So there exists a positive integer m such that $m_n \notin \{m_{i,j} : i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}\}$ for every $n \geq m$. Now pick $x^* \in S_{X^*}$ and, for each $i \in \{1, \dots, k\}$, we define $M_i := \{0, m_n\} \cup \bigcup_{j=1}^{n_i} \{m_{i,j}\}$ and $F_i, G_i : M_i \rightarrow X^*$ by the equations

$$F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}) \quad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$$

and

$$F_i(m_n) = -G_i(m_n) = d(m_n, 0)x^*, F_i(0) = G_i(0) = 0.$$

Now, for each unbounded or not uniformly discrete metric space M we have defined the desired subspaces M_i and functions F_i and G_i in $Lip(M_i, X^*)$ for every $1 \leq i \leq k$.

For a second step we claim that $\|F_i\|_{Lip(M_i, X^*)} \leq 1 + \delta$ for all $i \in \{1, \dots, k\}$. For this we have three cases again: M is unbounded, M is bounded, discrete but not uniformly discrete or M is bounded and $0 \in M'$.

Assume that M is unbounded. Given $u, v \in F, u \neq v$ and $i \in \{1, \dots, k\}$ we have two different possibilities:

a) If $u, v \notin \{m_N\}$ then

$$\frac{\|F_i(u) - F_i(v)\|}{d(u, v)} = \frac{\|g_i(u) - g_i(v)\|}{d(u, v)} \leq \|g_i\| \leq 1.$$

b) If $u = m_N$ then

$$\begin{aligned} \frac{\|F_i(u) - F_i(v)\|}{d(u, v)} &= \frac{\|d(m_N, 0)x^* - F_i(v)\|}{d(m_N, v)} \leq \frac{d(m_N, 0)}{d(m_N, v)} + \frac{\|g_i(v)\|}{d(m_N, v)} \leq \\ &\leq 1 + \frac{d(v, 0)}{d(m_N, v)} + \frac{\|g_i(v)\|}{d(m_N, v)} \stackrel{(2.1)}{<} 1 + \delta. \end{aligned}$$

Now, taking supremum in u and v , one has

$$\|F_i\|_{Lip(M_i, X^*)} \leq 1 + \delta.$$

Assume now that M is bounded, discrete but not uniformly discrete. Again given $u, v \in M_i, u \neq v$ and $i \in \{1, \dots, k\}$ we have different possibilities:

a) If $u \neq y_n$ and $v \neq y_n$ then we have

$$\frac{\|F_i(u) - F_i(v)\|}{d(u, v)} = \frac{\|g_i(u) - g_i(v)\|}{d(u, v)} \leq \|g_i\| \leq 1.$$

b) If $u = y_n, v \neq x_n$ then

$$\begin{aligned} \frac{\|F_i(u) - F_i(v)\|}{d(u, v)} &= \frac{\|g_i(x_n) + d(x_n, y_n)x^* - g_i(v)\|}{d(y_n, v)} \leq \frac{\|g_i(x_n) - g_i(v)\| + d(x_n, y_n)}{d(y_n, v)} \leq \\ &< \frac{d(x_n, v) + d(x_n, y_n)}{d(x_n, v) - d(y_n, x_n)} = \frac{1 + \frac{d(x_n, y_n)}{d(x_n, v)}}{1 - \frac{d(x_n, y_n)}{d(x_n, v)}} < 1 + \delta. \end{aligned}$$

c) If $u = y_n$ and $v = x_n$ then

$$\frac{\|F_i(u) - G_i(v)\|}{d(u, v)} = \frac{d(x_n, y_n)\|x^*\|}{d(x_n, y_n)} = 1$$

Then, taking supremum in u and v , one has

$$\|F_i\|_{Lip(M_i, X^*)} \leq 1 + \delta.$$

If M is bounded and $0 \in M'$ we can get also that

$$\|F_i\|_{Lip(M_i, X^*)} \leq 1 + \delta$$

using similar arguments to the ones of the above case taking n large enough.

Similar computations also arise

$$\|G_i\|_{Lip(M_i, X^*)} \leq 1 + \delta \quad \forall i \in \{1, \dots, k\}.$$

Now, we have defined subspaces $M_i \subset M$ and functions $F_i, G_i \in Lip(M_i, X^*)$ such that

$$\max_{1 \leq i \leq k} \{\|F_i\|_{Lip(M_i, X^*)}, \|G_i\|_{Lip(M_i, X^*)}\} \leq 1 + \delta.$$

As the pair (M, X^*) has the CPE then, for each $i \in \{1, \dots, k\}$, we can find an extension of F_i and G_i to the whole M respectively, which we will call again F_i and G_i , respectively, such that

$$\|F_i\|_{Lip(M, X^*)} \leq 1 + \delta, \|G_i\|_{Lip(M, X^*)} \leq 1 + \delta.$$

So $\frac{F_i}{1+\delta}, \frac{G_i}{1+\delta} \in B_{Lip(M, X^*)}$ for each $i \in \{1, \dots, k\}$.

The final step of the proof is to see that $\sum_{i=1}^k \frac{F_i}{1+\delta} \in C, \sum_{i=1}^k \frac{G_i}{1+\delta} \in C$ and to conclude from here that C has diameter 2. We prove this fact in the case M is unbounded. For the other cases, the arguments and estimates are similar. Then, assume M is unbounded. Given $i \in \{1, \dots, k\}$ one has

$$\varphi_i \left(\frac{F_i}{1+\delta} \right) = \frac{\sum_{j=1}^{n_i} \lambda_j^i F_i(m_{i,j})(x_{i,j})}{1+\delta} = \frac{\sum_{j=1}^{n_i} \lambda_j^i g_i(m_{i,j})(x_{i,j})}{1+\delta} = \frac{g_i(\varphi_i)}{1+\delta} > 1-\alpha.$$

So $\sum_{i=1}^k \lambda_i \frac{F_i}{1+\delta} \in C$. Similarly one has $\sum_{i=1}^k \lambda_i \frac{G_i}{1+\delta} \in C$. Hence

$$\begin{aligned} diam(C) &\geq \left\| \sum_{i=1}^k \lambda_i \frac{F_i}{1+\delta} - \sum_{i=1}^k \lambda_i \frac{G_i}{1+\delta} \right\| \geq \frac{\left\| \sum_{i=1}^k \lambda_i \frac{F_i(m_N)}{1+\delta} - \sum_{i=1}^k \lambda_i \frac{G_i(m_N)}{1+\delta} \right\|}{d(m_N, 0)} \\ &= \frac{\left\| \sum_{i=1}^k 2\lambda_i \frac{d(m_N, 0)x^*}{1+\delta} \right\|}{d(m_N, 0)} = \frac{2}{1+\delta}. \end{aligned}$$

From the estimate above we deduce that $\text{diam}(C) = 2$ from the arbitrariness of $0 < \delta < \delta_0$. ■

Now let us end the section by analysing the vector-valued Lipschitz-free Banach space over a concrete metric space. From here, we will get two interesting consequences: on the one hand, we will get several examples of vector-valued Lipschitz-free Banach spaces which not only fail to have an octahedral norm but also its unit ball contains points of Fréchet differentiability. On the other hand, we will prove that such construction depends strongly on the underlying target Banach space. So, octahedrality in vector-valued Lipschitz-free Banach spaces relies on the underlying metric spaces as well as on the target Banach one.

For the construction of such a metric space consider Γ to be an infinite set. Define $M := \Gamma \cup \{0\} \cup \{z\}$. Consider on M the following distance:

$$d(x, y) := \begin{cases} 1 & \text{if } x, y \in \Gamma \cup \{0\}, x \neq y, \\ 1 & \text{if } x = z, y \in \Gamma \text{ or } x \in \Gamma, y = z, \\ 2 & \text{if } x = z, y = 0 \text{ or } x = 0, y = z, \\ 0 & \text{Otherwise.} \end{cases}$$

This is obviously an infinite, bounded and uniformly discrete metric space. Moreover, it is not difficult to prove that the pair (M, X) has the CPE for every Banach space X . Consider a Banach space X , pick $y \in S_X$ and notice that $\delta_{z,y}$ is a 2-norm functional, so define $\varphi := \frac{\delta_{z,y}}{2} \in S_{\mathcal{F}(M,X)}$. Given $\alpha \in \mathbb{R}^+$ consider

$$S_\alpha := S\left(B_{\text{Lip}(M, X^*)}, \varphi, \frac{\alpha}{2}\right) = \{f \in B_{\text{Lip}(M, X^*)} / f(z)(y) > 2 - \alpha\}.$$

Consider $x \in \Gamma$ and $f \in S_\alpha$. We claim that

$$f(x)(y) > 1 - \alpha.$$

Indeed, assume by contradiction that $f(x)(y) \leq 1 - \alpha$. Then

$$1 < f(z)(y) - f(x)(y) = (f(z) - f(x))(y) \leq \|f(z) - f(x)\| \leq d(z, x) = 1,$$

a contradiction.

We will prove that $\inf_\alpha \text{diam}(S_\alpha)$ depends on the target space X^* .

Proposition 2.5. *If y is a point of Fréchet differentiability of B_X , then $\inf_\alpha S_\alpha = 0$.*

Proof. Notice that, as y is a point of Fréchet differentiability, then there exists (Smulian lemma) $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ and such that

$$(2.3) \quad \left. \begin{array}{l} x^*, y^* \in B_{X^*} \\ x^*(y) > 1 - \alpha \\ y^*(y) > 1 - \alpha \end{array} \right\} \Rightarrow \|x^* - y^*\| < \delta(\alpha).$$

Pick $f, g \in S(B_{Lip(M, X^*)}, \varphi, \frac{\alpha}{2})$ and $u, v \in M \setminus \{0\}, u \neq v$. Our aim is to estimate

$$\begin{aligned} \frac{\|f(u) - g(u) - (f(v) - g(v))\|}{d(u, v)} &\leq \|f(u) - g(u) - (f(v) - g(v))\| \leq \\ &\leq \|f(u) - g(u)\| + \|f(v) - g(v)\| =: K. \end{aligned}$$

If $u = z$ then we have

$$\frac{f(u)(y)}{2} > 1 - \frac{\alpha}{2}, \frac{g(u)(y)}{2} > 1 - \frac{\alpha}{2} \xrightarrow{(2.3)} \|f(u) - g(u)\| \leq 2\delta\left(\frac{\alpha}{2}\right).$$

Similarly, if $u \in \Gamma$ then

$$f(u)(y) > 1 - \alpha, g(u)(y) > 1 - \alpha \xrightarrow{(2.3)} \|f(u) - g(u)\| \leq \delta(\alpha).$$

Hence $K \leq \delta(\alpha) + \max\left\{\delta(\alpha), 2\delta\left(\frac{\alpha}{2}\right)\right\}$.

From the arbitrariness of $f, g \in S(B_{Lip(M)}, \varphi, \frac{\alpha}{2})$ we conclude that

$$\text{diam}\left(S\left(B_{Lip(M)}, \varphi, \frac{\alpha}{2}\right)\right) \leq \delta(\alpha) + \max\left\{\delta(\alpha), 2\delta\left(\frac{\alpha}{2}\right)\right\}.$$

Finally, taking infimum in $\alpha \in \mathbb{R}^+$, from the hypothesis on δ and the continuity of the map \max we conclude the desired result. ■

Despite above Proposition, we will prove that $\mathcal{F}(M, X)$ has a dramatically different behaviour whenever X^* has the weak-star slice diameter two property.

Proposition 2.6. *If X^* has the weak-star slice diameter two property, then $\inf_{\alpha} S_{\alpha} = 2$.*

Proof. Pick two arbitrary numbers $\alpha > 0$ and $\varepsilon > 0$. As X^* has the weak-star slice diameter two property we can find $x^*, y^* \in S(B_{X^*}, x, \frac{\alpha}{2})$ such that $\|x^* - y^*\| > 2 - \varepsilon$. Now define $f, g : M \rightarrow X^*$ by the equations

$$f(t) := d(t, 0)x^* \quad g(t) := d(t, 0)y^* \quad \forall t \in M.$$

Now f, g are clearly norm one Lipschitz functions. Moreover

$$\varphi(f) = \frac{f(z)(x)}{2} = x^*(x) > 1 - \frac{\alpha}{2}.$$

So $f \in S_{\alpha}$. Analogously $g \in S_{\alpha}$. Consequently

$$\text{diam}(S_{\alpha}) \geq \|f - g\| \geq \frac{\|f(z) - g(z)\|}{2} = \|x^* - y^*\| > 2 - \varepsilon.$$

As ε and α were arbitrary we conclude that $\text{diam}(S_{\alpha}) = 2$, so we are done. ■

From two Propositions above we can get the desired consequences. From Proposition 2.5 we get vector-valued Lipschitz-free Banach spaces with points of Fréchet differentiability which, keeping in mind that the pair (M, X^*) has the (CPE) for every Banach space X , proves that Theorem 2.4 is optimal. However, from Proposition 2.6 we conclude that the existence of such Fréchet

differentiability point depends on the target space. Indeed, we can even get octahedrality for suitable choices of X in example above. For instance, $\mathcal{F}(M, \ell_1) = \mathcal{F}(M) \widehat{\otimes}_\pi \ell_1 = \ell_1(\mathcal{F}(M))$ has an octahedral norm.

3. CONSEQUENCES AND OPEN QUESTIONS.

Under the assumptions of Theorem 2.4 we have that $Lip(M, X^*)$ has the weak-star strong diameter two property. This arises a natural question.

Question 1. *Let M and X under the hypothesis of Theorem 2.4. Does $Lip(M, X^*)$ satisfy the strong diameter two property?*

Note that we have a partial answer in [17] for the scalar case in terms of Daugavet property. Also, in the scalar case, when M is a compact metric space such that $lip(M)$ separates the points in M , it is known that $lip(M)^* = \mathcal{F}(M)$ and $lip(M)$ is an M -embedded space (see Remark after Theorem 6.6 in [19]), that is $lip(M)$ is an M -ideal in $Lip(M)$ (see [7] for the case $M = [0, 1]$). Then we get from [1] that $lip(M)$ and $Lip(M)$ satisfy the strong diameter two property. Recall that $lip(M)$ stands for the space of scalar Lipschitz functions on M such that

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)}, x \neq y \in M, d(x, y) < \varepsilon \right\} = 0.$$

Moreover, in [16, Theorems 1 and 2] the author get in the scalar case that $Lip(M)$ has the slice diameter two property whenever M satisfy the same assumptions to the ones of Theorem 2.4.

In [9] it has been recently proved that $\mathcal{F}(M)$ contains an isomorphic copy of ℓ_1 whenever M is an infinite metric space. However, this fact is an easy consequence of Theorem 2.4.

Corollary 3.1. *Let M be an infinite metric space with a designated origin 0. Then $\mathcal{F}(M)$ contains an isomorphic copy of ℓ_1 .*

Proof. The corollary follows whenever M satisfies any of the assumptions of Theorem 2.4.

In other case, the corollary follows from [13, Proposition 5.1]. ■

Remark 3.2. In [9] is proved that $\mathcal{F}(M)$ contains a complemented copy of ℓ_1 whenever M is an infinite metric space. In fact, the results in [9] give that $\mathcal{F}(M, X)$ contains too a complemented copy of ℓ_1 , for every Banach space X . Indeed, as $Lip(M)$ is isometrically isomorphic to a closed subspace of $Lip(M, X)$, then $Lip(M, X)$ contains an isomorphic copy of ℓ_∞ . Finally, from [8, Th. 4], we obtain that $\mathcal{F}(M, X)$ contains a complemented copy of ℓ_1 . We want to thank M. Doucha for noticing us about this fact. We would also want to thank M. Cúth for asking us about the identification of vector-valued Lipschitz-free Banach spaces with a projective tensor product spaces.

Propositions 2.5 and 2.6 show that geometry of vector-valued Lipschitz-free Banach spaces does not only depend on underlying scalar Lipschitz-free space but also on the target Banach space. However, two natural questions arise.

Question 2. *Let M be a pointed metric space and let X be a non-zero Banach space.*

- (1) *Does Theorem 2.4 hold without assuming that the pair (M, X^*) has the CPE?*
- (2) *Does $\mathcal{F}(M, X)$ have an octahedral norm whenever $\mathcal{F}(M)$ does?*

Bearing in mind the identification $\mathcal{F}(M, X) = \mathcal{F}(M) \widehat{\otimes}_\pi X$, above Question is related to the problem of how octahedrality is preserved by projective tensor products. However, the last one is an open problem recently posed in [20].

Finally, we have analysed octahedrality in $\mathcal{F}(M, X)$ whenever M is a metric space and X is a Banach space. However, we did not get any result about the dual properties (i.e. diameter two properties). More strictly.

Question 3. *Given M a metric space and X a non-zero Banach space.*

Which assumptions do we need over M and X in order to ensure that $\mathcal{F}(M, X)$ has the slice diameter two property (respectively diameter two property, strong diameter two property)?

Again, not only do we get a partial answer in scalar case but also in vector valued one. Indeed, again by [17] we know that $\mathcal{F}(M)$ has the strong diameter two property whenever M is a metrically convex metric space. Keeping in mind that $\mathcal{F}(M, X) = \mathcal{F}(M) \widehat{\otimes}_\pi X$, next Proposition is an immediate application of [5, Corollary 3.6].

Proposition 3.3. *Let M be a metric space with a designated origin 0 and X be a Banach space. If M is metrically convex and X has the strong diameter two property, then $\mathcal{F}(M, X)$ has the strong diameter two property.*

Despite above Proposition, there are metric spaces whose free-Lipschitz Banach space fails to have any diameter two property. Indeed, it is well known that $\mathcal{F}(M)$ has the Radon-Nikodym property whenever M is a totally discrete metric sapce [19]. Related to the strong diameter two property we can even get vector valued free Lipschitz Banach spaces which fail such property. Indeed, if we consider X a Banach space failing the strong diameter two property and M a totally discrete metric space then $\mathcal{F}(M, X) = \mathcal{F}(M) \widehat{\otimes}_\pi X$ does not have the strong diameter two property [5, Corollary 3.13].

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